Development of Confined Stokes-Flow Wakes

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A general solution is presented for the development of the Stokes-flow wake behind a cascade-like stack of blunt-based bodies in steady two-dimensional laminar incompressible flow. The solution is obtained as a Fourier series expansion in the transverse coordinate whose coefficients are given in closed form as functions of the velocity field in the plane of the base. Details are presented for the wake behind an infinite stack of finite thickness flat plates and it is seen that the existence of a recirculation region behind the base is intimately connected to the base plane transverse velocity component.

Introduction

THERE has been much recent interest in the study of steady two-dimensional laminar incompressible separated flows with closed streamlines. For the most part, fundamental studies consider relatively simple geometries such as those occurring in wake and channel flows and are concerned with the solution of equations that approximate the Navier-Stokes equations for various Reynolds number limits.

Channel flow solutions with boundary-layer-like approximations for large Reynolds numbers appear in Kumar and Yajnik,[1] Plotkin,[2] and Acrivos and Schrader.[3] Scarpi[4] considers a Stokes-flow approximation for small Reynolds numbers. Plotkin[5] studied the wake flow behind a stack of blunt-based bodies for large Reynolds numbers using the slender-channel equations and a Fourier series expansion technique. The base- and near-wake flow problem for very low Reynolds numbers was studied extensively in the 1960s and much of this research is discussed in Ref. 6.

Viviand and Berger[7] derive a general solution for the development of the Stokes-flow wake behind a blunt-based body in an unbounded fluid. The solution is obtained in terms of Poisson integrals whose integrands contain the velocity components in the plane of the base. It is shown that a recirculation region can exist behind the base for positive values of the transverse velocity component in the base plane.

The main goal of the present research is to derive a general solution for the development of the Stokes-flow wake behind a cascade-like stack of blunt-based bodies in steady two-dimensional laminar incompressible flow. This is equivalent to the problem studied by Viviand and Berger[7] with the addition of periodic boundary conditions in the transverse direction. An additional goal of the research is to study further the convergence properties of the Fourier series expansion (spectral, Galerkin) method that was used by Plotkin[2,5] to solve the nonlinear, large Reynolds number, slender-channel equations. Finally, the solution obtained from this research will provide an accurate test case for numerical solution techniques developed to calculate separated flows.

Problem Formulation and Method of Solution

Consider the steady two-dimensional laminar incompressible flow past a symmetric cascade of blunt-based bodies as illustrated by the finite thickness flat plates shown in Fig. 1. The x axis is taken perpendicular to the base plane with its origin at the base center. It is an axis of flow symmetry. x and y are nondimensionalized by half of the separation between the plate centerlines and the velocities are nondimensionalized by the speed of the downstream uniform stream. h is the ratio of plate thickness to plate separation and appears in the problem in the specification of the velocity field at the base plane.

The stream function Ψ is introduced such that

$$u=\Psi_y, \qquad v=-\Psi_x \tag{1}$$

where u and v are the velocity components in the x and y directions, respectively. The Navier-Stokes equation is

$$R(\Psi_y\nabla^2\Psi_x-\Psi_x\nabla^2\Psi_y)=\nabla^4\Psi \tag{2}$$

where the Reynolds number R is based on the characteristic length and speed used above. The Stokes-flow (zero Reynolds number) limit of Eq. (1) is

$$\nabla^4\Psi=0 \tag{3}$$

The transverse boundaries are symmetry boundaries with the following boundary conditions:

$$\Psi(x,\pm 1)=\pm 1 \tag{4a}$$

$$\Psi_{yy}(x,\pm 1)=0 \tag{4b}$$

The elliptic Stokes-flow problem requires that boundary conditions be specified at upstream infinity. In this paper, the wake development downstream of the base plane is sought given an arbitrary but reasonable choice of base plane initial profiles. These might, for example, be curve fits to experimentally determined initial profiles. The velocity field is given at the base plane

$$\Psi_y(0,y)=u_i(y) \tag{5a}$$

$$-\Psi_x(0,y)=v_i(y) \tag{5b}$$

and the uniform stream is recovered far downstream

$$\Psi(\infty,y)=y \tag{6}$$

The Fourier series expansion method used by Plotkin[5] for the large Reynolds number version of the problem is introduced to identically satisfy Eqs. (4),

$$\Psi=y+\sum_{1}^{N}a_n(x)\sin n\pi y \tag{7}$$

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where the series is truncated after N times. This is a spectral[8] or Galerkin method. Equation (7) is substituted into Eq. (3), the result is multiplied by $\sin m\pi y$ and integrated across the flow, and the resulting linear set of ordinary differential equations is obtained,

$$a_m'''' - 2m^2\pi^2 a_m'' + m^4\pi^4 a_m = 0 \qquad m = 1,N \tag{8}$$

The general solution of Eq. (8) is

$$a_m(x) = (A_m + xB_m)\exp(-m\pi x) + (C_m + xD_m)\exp(m\pi x) \tag{9}$$

where A_m, B_m, C_m, and D_m are constants. Note[7] that the solution is of the form

$$\Psi = y + V_1 + xV_2 \tag{10}$$

where V_1 and V_2 each satisfy Laplace's equation.

The infinity condition [Eq. (6)] yields $C_m = D_m = 0$. The base plane boundary conditions [Eqs. (5)] lead to the following conditions on the Fourier coefficients:

$$a_{mi} \equiv a_m(0) = \frac{2}{\pi m}\int_0^1 u(y)\cos m\pi y \, dy \tag{11a}$$

$$a'_{mi} \equiv a'_m(0) = -2\int_0^1 v_i(y)\sin m\pi y \, dy \tag{11b}$$

and with the use of Eq. (9) the unknown constants become

$$A_m = a_{mi} \tag{12a}$$

$$B_m = m\pi a_{mi} + a'_{mi} \tag{12b}$$

The Fourier (spectral) coefficients $a_m(x)$ are therefore obtained in closed form.

The pressure distribution can be found following the approach in Viviand and Berger.[7] Introduce the Stokes-flow pressure $\bar{p}$ and vorticity Ω where

$$\bar{p} = Rp \tag{13a}$$

$$\Omega = v_x - u_y = -\nabla^2\Psi \tag{13b}$$

and p is nondimensionalized by density times speed squared. The x and y momentum equations can be written

$$\bar{p}_x = -\Omega_y, \qquad \bar{p}_y = \Omega_x \tag{14}$$

and, with the use of Eqs. (10) and (13b),

$$\Omega = -2V_{2x} \tag{15}$$

Eqs. (14) and (15) can then be solved to yield

$$\bar{p} - \bar{p}_\infty = V_{2y} = 2\sum_1^N m\pi B_m \cos m\pi y \exp(-m\pi x) \tag{16}$$

where $\bar{p}_\infty$ is the pressure in the stream.

Results and Discussion

Equations (7) and (9-12) yield the solution for the stream function in terms of an arbitrary velocity field at the base plane. This allows for a calculation of the wake development downstream of the base. In general, for a specific geometry, the velocity at $x = 0$ is not known in advance.

In what follows, the calculations will be presented for an assumed "reasonable" pair of velocity components compatible with the finite thickness plate cascade of Fig. 1. The analysis will be guided by the results of Viviand and Berger[7] for the wake downstream of an isolated base.

The flow upstream of the base is assumed to be fully developed and the streamwise component of velocity is taken to be the Poiseuille parabolic profile

$$u_i = \frac{3}{1-h}\left[\frac{y-h}{1-h} - \frac{1}{2}\left(\frac{y-h}{1-h}\right)^2\right] \qquad h \le y \le 1$$
$$= 0 \qquad 0 \le y \le h \tag{17}$$

with $u_i(-y) = u_i(y)$. This is identical to the profile used in Plotkin.[5] From Eq. (11a),

$$a_{mi} = 6[\sin m\pi h + m\pi(1-h)\cos m\pi h]/\pi^4 m^4 (h-1)^3 \tag{18}$$

In general, $v_i(y) \ne 0$ due to the upstream influence of the base. Viviand and Berger[7] suggest a choice of v_i such that $v_i(h) = v_i(1) = 0$, $v_i(-y) = -v_i(y)$, and either $v_i \le 0$ or $v_i \ge 0$ for

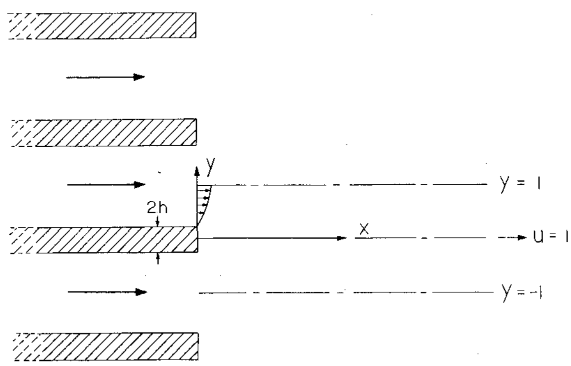

Fig. 1 Flow configuration and coordinate system.

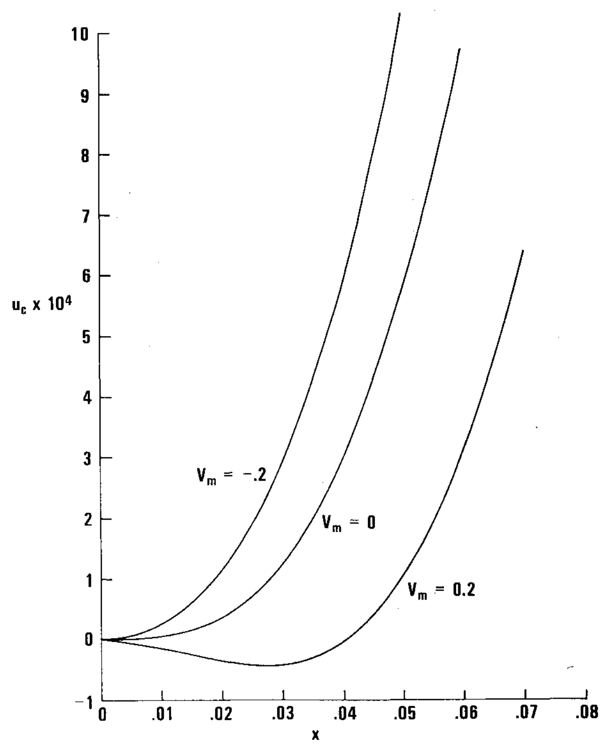

Fig. 2 Wake centerline velocity for $h = 0.5$.

$h \le y \le 1$. The following parabolic profile is chosen:

$$v_i = V_m\left[1-\frac{4}{(1-h)^2}\left(y-\frac{1+h}{2}\right)^2\right] \qquad h \le y \le 1$$

$$= 0 \qquad 0 \le y \le h \qquad (19)$$

and from Eq. (11b),

$$a'_{mi} = \frac{-8V_m}{m^3\pi^3(1-h)^2}[2\cos m\pi h - 2\cos m\pi + m\pi(h-1)\sin m\pi h] \qquad (20)$$

The calculated solutions are qualitatively identical to those of Viviand and Berger.[7] For $V_m \le 0$, no recirculation region appears behind the base. For $V_m > 0$, a recirculation region appears behind the base and its length is found to be proportional to V_m. The magnitude of the reverse flow velocity is extremely small. In the results that follow, the values of V_m are chosen to be 0 and ± 0.2.

The wake centerline velocity is obtained from Eqs. (1) and (7) as

$$u_c = 1 + \sum_{1}^{N} n\pi a_n(x) \qquad (21)$$

The length of the recirculation region x_r is determined from

$$u_c(x_r) = 0 \qquad (22)$$

The wake centerline velocity is shown in Fig. 2 for the three values of V_m and $h=0.5$. Note the region of reverse flow for $V_m = 0.2$ and also note that the maximum value of the reverse flow component is of the order of 10^{-4} times the stream velocity. The streamline pattern for $V_m = 0$ and $h = 0.5$ is shown in Fig. 3. There is no recirculation region and the streamlines are seen to conform to the base. The streamline pattern for $V_m = 0.2$ and $h = 0.5$ is shown in Fig. 4 and the recirculation bubble can be clearly seen. Note the small values of the stream function on the closed streamlines and note also that the length of the bubble is less than 10% of half of the base height.

The wake centerline velocity for $V_m = 0.2$ is shown as a function of h in Fig. 5. The case $h=0$ represents a stack of flat plates and does not possess a recirculation bubble. The length of the recirculation region for $V_m = 0.2$ is shown as a function of h in Fig. 6. In a related problem, Acrivos and Schrader[3] suggest that a finite limit exists for x_r/h as $h \to 0$. To see if that is the case here, x_r/h has been plotted vs h^{-1} (note the log scale) and it does appear that a finite limit exists.

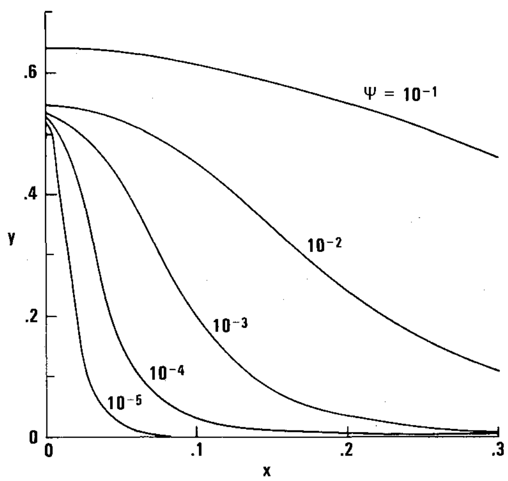

Fig. 3 Streamline pattern for $V_m = 0$ and $h = 0.5$.

To obtain the numerical values plotted in Figs. 2-6, the Fourier series expansion in Eq. (6) or its derivatives was summed to convergence. Convergence to at least eight significant figures could be obtained for any value of $x > 0$. It was found that in general hundreds of terms were needed for convergence to three significant figures for points inside the recirculation region, but that the number of terms dropped significantly once $x > x_r$. These results suggest that the spectral method does not appear to be an efficient technique to numerically integrate the Navier-Stokes equations for these geometries for finite but very low Reynolds numbers.

Viviand and Berger[7] chose the streamwise velocity component at the base plane u_i such that

$$u_{i_y}(h) = 0$$

This was necessary to prevent the occurrence of a divergent integral in the calculation of the base pressure. Note that u_i from Eq. (17) does not satisfy the above corner separation condition. The discontinuity in u_i keeps the Fourier series

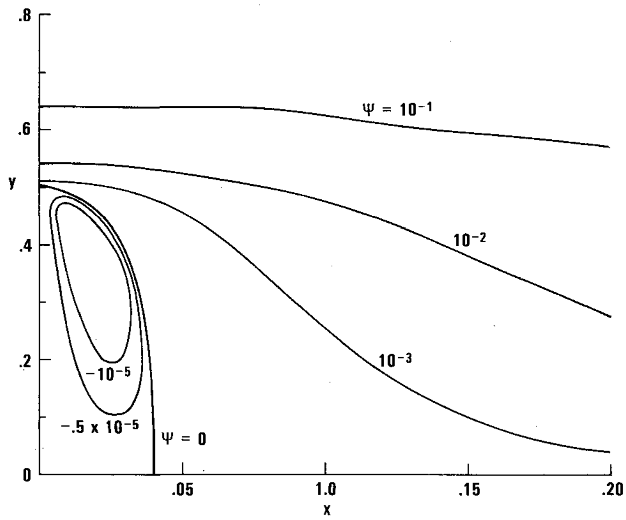

Fig. 4 Streamline pattern for $V_m = 0.2$ and $h = 0.5$.

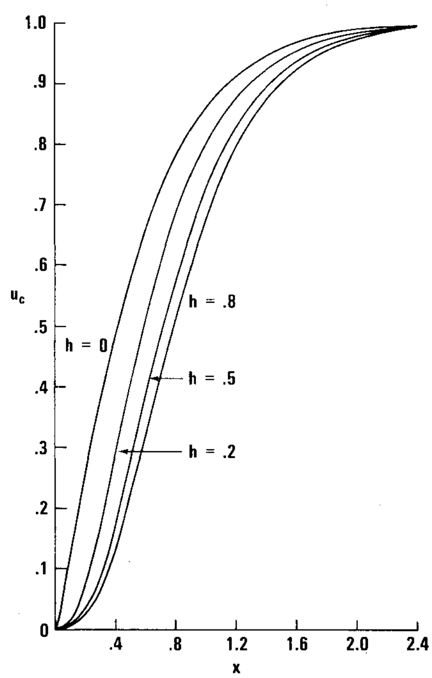

Fig. 5 Wake centerline velocity for $V_m = 0.2$ as a function of ratio of plate thickness to plate separation h.

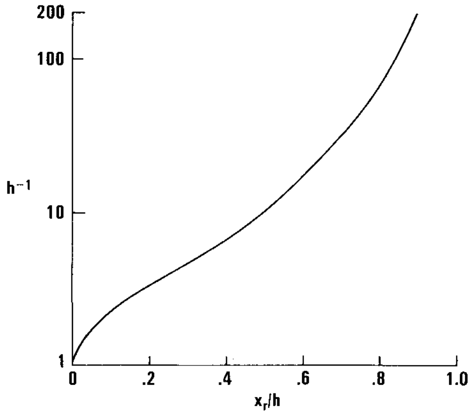

Fig. 6 Length of recirculation region x_r for $V_m = 0.2$ as a function of h.

from convergence at $x=0$ so that the base pressure cannot be calculated from Eq. (16) at $x=0$ for any choice of u_i for $h \neq 0$. However, pressure calculations with Eq. (16) for $x>0$ demonstrate that the pressure is continuous at $x=0$ so that the base pressure can be obtained.

In the large Reynolds number calculations of Plotkin,[5] it was shown that u_i from Eq. (17) is an appropriate choice and that the use of a "separating" profile leads to order of magnitude changes in the resulting flowfield. Consider now the "separating" profile

$$u_i = \frac{1}{1-h}\left[6\left(\frac{y-h}{1-h}\right)^2 - 4\left(\frac{y-h}{1-h}\right)^3\right] \quad h \leq y \leq 1$$

$$= 0 \qquad 0 \leq y \leq h \qquad (23)$$

with

$$a_{mi} = 24[2\cos m\pi - 2\cos m\pi h + m\pi(1-h)\sin m\pi h]/m^5\pi^5(1-h)^4 \qquad (24)$$

A calculation was made with the v_i of Eq. (19) with $V_m = 0.2$ and $h = 0.5$. The resulting flowfield was found to be both qualitatively and quantitatively similar to the previous results. The reverse flow velocity is still of the order of 10^{-4} times the stream velocity and x_r changes from approximately 0.041 to only 0.050.

If one were to solve the problem numerically, or if one were to use the spectral method approach to solve the Navier-Stokes equations for small but finite Reynolds numbers, downstream boundary conditions would need to be satisfied at some outflow boundary $x = x_0$. Two questions immediately arise:

1) How does one choose the location of x_0?

2) What boundary conditions should be satisfied at x_0?

Some insight into the answers to these questions can be gained by a study of the present exact Stokes-flow solution with outflow boundary conditions added.

A discussion of common outflow boundary conditions is given in Roache.[9] Many are of the continuation type- streamwise derivatives of the dependent variables are set equal to zero. Consider first the choice

$$a_m'(x_0) = a_m''(x_0) = 0 \qquad (25)$$

which is equivalent physically to setting $v = v_x = 0$ at x_0. Equations (11) and (25) are applied to the general solution for a_m in Eq. (9) to yield a set of linear algebraic equations whose solution yields the constants A_m, B_m, C_m, and D_m as functions of x_0. Figure 7 shows the wake centerline velocity in the recirculation region ($h = 0.5$, $V_m = 0.2$) as a function of x_0. The exact

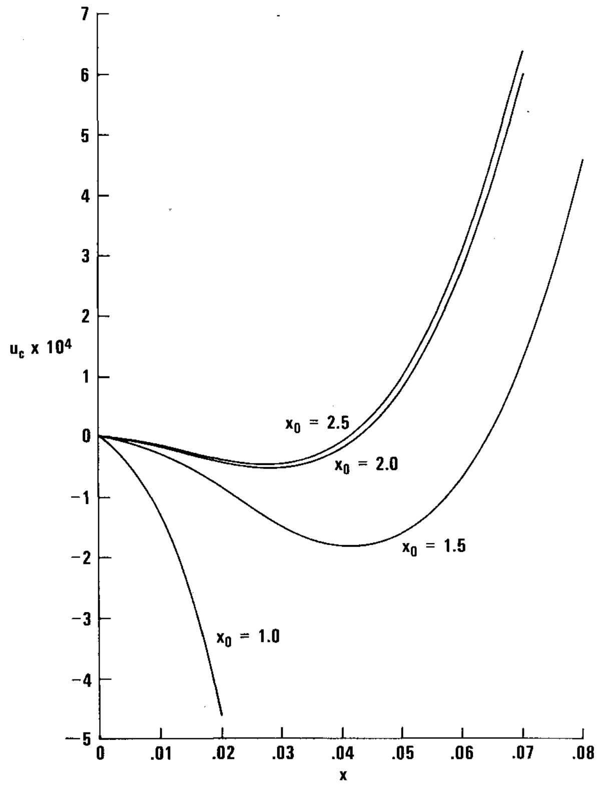

Fig. 7 Wake centerline velocity for $V_m = 0.2$ and $h = 0.5$ as a function of location of outflow boundary x_0.

solution would lie on top of the $x_0 = 2.5$ curve, but three significant figure accuracy for a calculation with $\Delta x = 0.01$ is first achieved for all points for $x_0 = 3.5$. The calculations were repeated with $h = 0.2$ and three significant figure accuracy again was first achieved with $x_0 = 3.5$. Therefore, it appears that for this problem the choice of x_0 is dependent on the plate separation and not the plate thickness.

The above exercise was repeated for the stronger outflow boundary conditions

$$a_m(x_0) = a_m'(x_0) = 0 \qquad (26)$$

which correspond physically to setting $u = v = 0$ at x_0. In general, it is not recommended to apply the "exact" infinity conditions at a finite downstream boundary. It was observed that with the use of Eq. (26), a recirculation region did not appear in the calculations until $x_0 \geq 2$, which was not the case for Eq. (25) as is shown in Fig. 7. However, three significant figure accuracy for Eq. (26) was also first achieved for $x_0 = 3.5$.

Conclusions

A general solution has been derived for the development of the Stokes-flow wake behind a cascade-like stack of blunt-based bodies in steady two-dimensional laminar incompressible flow. The effect of the choice of inflow and outflow boundary conditions on the solution has been studied and it is seen that a recirculation region behind the base is present only for positive values of the base plane transverse velocity component. The convergence properties of the Fourier series expansion (spectral) method have been studied and it appears that the technique would not be efficient for the numerical integration of the full Navier-Stokes equations for this geometry for finite but small values of the Reynolds number. Finally, the solution provides one of only a few appropriate examples to test the accuracy of numerical solution techniques developed to calculate laminar flows with separation.

Acknowledgment

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References

[1]Kumar, A. and Yajnik, K. S., "Internal Separated Flows at Large Reynolds Number," *Journal of Fluid Mechanics*, Vol. 97, Pt. 1, 1980, pp. 27-51.

[2]Plotkin, A., "Spectral Method Solutions for Some Laminar Channel Flows with Separation," *AIAA Journal*, Vol. 20, Dec. 1982, pp. 1713-1719.

[3]Acrivos, A. and Schrader, M. L., "Steady Flow in a Sudden Expansion at High Reynolds Numbers," *Physics of Fluids*, Vol. 25, June 1982, pp. 923-930.

[4]Scarpi, G., "The Development of a Plane Laminar Flow at Low Reynolds Numbers," *Meccanica*, Vol. 16, Dec. 1981, pp. 199-203.

[5]Plotkin, A., "Development of Confined Laminar Wakes at Large Reynolds Numbers," *AIAA Journal*, Vol. 20, Feb. 1982, pp. 211-217.

[6]Berger, S. A., *Laminar Wakes*, American Elsevier Publishing Co., New York, 1981, Chap. 8.

[7]Viviand, H. and Berger, S. A., "The Base-Flow and Near-Wake Problem at Very Low Reynolds Numbers, Part I: The Stokes Approximation," *Journal of Fluid Mechanics*, Vol. 23, Pt. 3, 1965, pp. 417-438.

[8]Gottlieb, D. and Orszag, S. A., *Numerical Analysis of Spectral Methods: Theory and Applications*, SIAM, Philadelphia, Pa., 1977.

[9]Roache, P. J., *Computational Fluid Dynamics*, Hermosa Publishers, Albuquerque, N. Mex., 1972, pp. 154-161.